Graph Theory
Homework 7

Joshua Ruiter

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Proposition 0.1 (Exercise 1a)**.** *Let $p \in [0,1]$, and $\mathcal{G}(n,P)$ be the set of graphs with n vertices with probability measure $\mathbb{P}(G) = p^m q^{N-m}$ where $q = 1-p, m = \text{edge}(G), N = \binom{n}{2}$. A fixed edge e has probability p of being present, $\mathbb{P}(e \in G) = p$.*

Proof. First, we write $\mathbb{P}(e \in G)$ as the sum over G with fixed m.

$$\mathbb{P}(e \in G) = \sum_{\substack{G \\ e \in G}} \mathbb{P}(G) = \sum_{m=0}^{N} \sum_{\substack{G \\ e\in G \\ \text{edge}(G)=m}} p^m q^{N-m}$$

There are $\binom{N}{m}$ graphs with $\text{edge}(G) = m$, but we want to count G with a fixed edge e, which gives us one less choice to make, so there are $\binom{N-1}{m-1}$ graphs with $\text{edge}(G) = m$ containing e.

$$\sum_{m=0}^{N} \sum_{\substack{G \\ e\in G \\ \text{edge}(G)=m}} p^m q^{N-m} = \sum_{m=0}^{N} \binom{N-1}{m-1} p^m q^{N-m}$$

Now we use Pascal's identity $\binom{N}{m} = \binom{N-1}{m-1} + \binom{N-1}{m}$ and the Binomial Theorem to simplify.

$$\begin{aligned}
\sum_{m=0}^{N} \binom{N-1}{m-1} p^m q^{N-m} &= \sum_{m=0}^{N} \left(\binom{N}{m} - \binom{N-1}{m} \right) p^m q^{N-m} \\
&= \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} - \sum_{m=0}^{N} \binom{N-1}{m} p^m q^{N-m} \\
&= \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} - q \sum_{m=0}^{N-1} \binom{N-1}{m} p^m q^{(N-1)-m} \\
&= (p+q)^N - q(p+q)^{N-1} \\
&= 1 - q \\
&= p
\end{aligned}$$

The upper index change from N to $N-1$ is valid because $\binom{N-1}{N} = 0$. $\square$

Proposition 0.2 (Exercise 1b)**.** *Let $\mathcal{G}(n,p)$ be as in part (a). The probability of two different edges being in G are independent events. That is,*

$$\mathbb{P}(e, e' \in G) = \mathbb{P}(e \in G)\mathbb{P}(e' \in G) = p^2$$

Proof. The proof is essentially the same as for part (a). We write $\mathbb{P}(e, e' \in G)$ as the sum over G with fixed m.

$$\mathbb{P}(e, e' \in G) = \sum_{\substack{G \\ e,e' \in G}} \mathbb{P}(G) = \sum_{m=0}^{N} \sum_{\substack{G \\ e,e' \in G \\ \text{edge}(G)=m}} p^m q^{N-m}$$

There are $\binom{N}{m}$ graphs with $\text{edge}(G) = m$, so there are $\binom{N-2}{m-2}$ graphs with $\text{edge}(G) = m$ containing e and e'.

$$\sum_{m=0}^{N} \sum_{\substack{G \\ e \in G \\ \text{edge}(G)=m}} p^m q^{N-m} = \sum_{m=0}^{N} \binom{N-2}{m-2} p^m q^{N-m}$$

Applying Pascal's identity three times, we obtain

$$\binom{N-2}{m-2} = \binom{N}{m} - 2\binom{N-1}{m} + \binom{N-2}{m}$$

Now we can simplify $\mathbb{P}(e, e' \in G)$, using the Binomial Theorem and the same upper-index change trick as in part (a).

$$\begin{aligned}
\mathbb{P}(e, e' \in G) &= \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} - 2\sum_{m=0}^{N} \binom{N-1}{m} p^m q^{N-m} + \sum_{m=0}^{N} \binom{N-2}{m} p^m q^{N-m} \\
&= \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} - 2q\sum_{m=0}^{N-1} \binom{N-1}{m} p^m q^{N-1-m} + q^2\sum_{m=0}^{N-2} \binom{N-2}{m} p^m q^{N-2-m} \\
&= (p+q)^N - 2q(p+q)^{N-1} + q^2(p+q)^{N-2} \\
&= 1 - 2q + q^2 \\
&= (q-1)^2 \\
&= p^2
\end{aligned}$$

□

Proposition 0.3 (Exercise 2a)**.** *Let $0 < p < 1$, and define $\alpha(G)$ to be the size of a maximum independent set in a graph G. For $G \in \mathcal{G}(n,p)$ and $s \in \mathbb{Z}_{\geq 0}$,*

$$\mathbb{P}(\alpha(G) \geq s) \leq \binom{n}{s}(1-p)^{\binom{s}{2}}$$

Proof.

$$\begin{aligned}
\mathbb{P}(\alpha(G) \geq s) &= \mathbb{P}\Big(\exists S = \{v_1, \ldots, v_k\} \subset G, k \geq s; v_i v_j \notin G, \forall 1 \leq i < j \leq k\}\Big) \\
&\leq \mathbb{P}\Big(\exists S = \{v_1, \ldots, v_s\} \subset G, v_i v_j \notin G, \forall 1 \leq i < j \leq s\Big) \\
&\leq \sum_{S=\{v_1,\ldots,v_s\}} \prod_{1 \leq i < j \leq s} \mathbb{P}(v_i v_j \notin G) \\
&= \binom{n}{s}(1-p)^{\binom{s}{2}}
\end{aligned}$$

Note that this is not an equality because there may be an independent set of size strictly greater than s, or there may be two different independent sets of size s. □

Proposition 0.4 (Exercise 2b)**.** *Let* $0 < p < 1$*, and define* $\mathrm{tri}(G)$ *to be the number of 3-cycles in* G*. For* $G \in \mathcal{G}(n,p)$*,*

$$\mathbb{P}\left(\mathrm{tri}(G) \geq \frac{n}{2}\right) \leq \frac{\mathbb{E}(\mathrm{tri})}{\frac{1}{2}n} = \frac{1}{3}(n-1)(n-2)p^3$$

Proof. The first inequality is just Markov's inequality. There are $\binom{n}{3}$ distinct 3-cycles in G. Label the cycles $c_1, \ldots, c_{\binom{n}{3}}$, and define

$$f_i(G) = \begin{cases} 1 & c_i \subset G \\ 0 & \text{else} \end{cases}$$

Then $\mathbb{E}(f_i) = \mathbb{P}(c_i \subset G) = p^3$, and

$$\mathbb{E}(\mathrm{tri}) = \sum_{i=1}^{\binom{n}{3}} \mathbb{E}(f_i) = \binom{n}{3}p^3$$

Now the equality follows.

$$\frac{\mathbb{E}(\mathrm{tri})}{\frac{1}{2}n} = \frac{\frac{1}{6}n(n-1)(n-2)p^3}{\frac{1}{2}n} = \frac{1}{3}(n-1)(n-2)p^3$$

□

Proposition 0.5 (Exercise 2c)**.** *Let* G *be a graph with* $|G| = n$*, and let* $s = \lceil \frac{n}{6} \rceil$*. Suppose* G *satisfies* $\alpha(G) < s$ *and* $\mathrm{tri}(G) < \frac{n}{2}$*. Then there is a subgraph* $H \subset G$ *with* $|H| \geq \frac{n}{2}$ *such that* $\chi(H) > 3$ *and* $\mathrm{girth}(H) > 3$*.*

Proof. From each 3-cycle in G, delete one vertex, and take H to be the remaining induced subgraph of G. Since we remove at most $n/2$ vertices, $|H| \geq \frac{n}{2}$. Clearly, $\mathrm{girth}(H) > 3$, since every 3-cycle in G was broken.

Note that for any graph H, $\chi(H)\alpha(H) \geq |H|$, since $\alpha(H)$ is the maximum size of an independent set and $\chi(H)$ is the minimum number of mutually disjoint independent sets. Also note that $\alpha(G) \geq \alpha(H)$. Thus

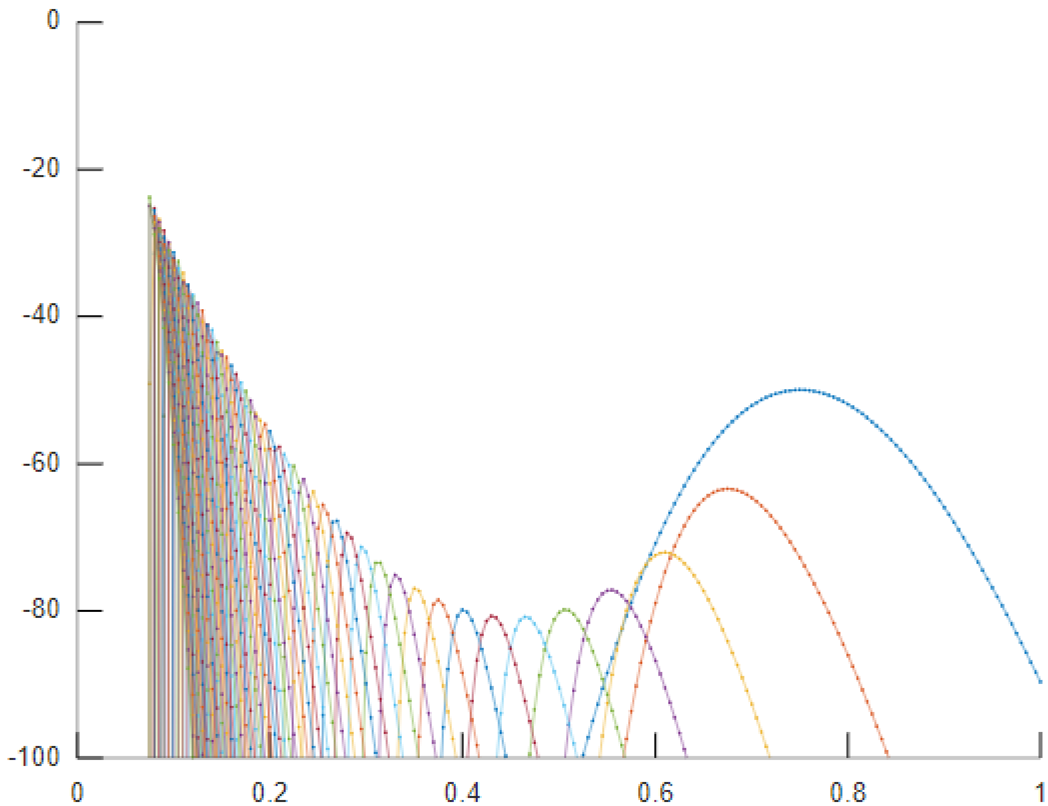
$$\chi(H) \geq \frac{|H|}{\alpha(H)} \geq \frac{n/2}{\alpha(G)} > \frac{n/2}{n/6} = 3$$

□

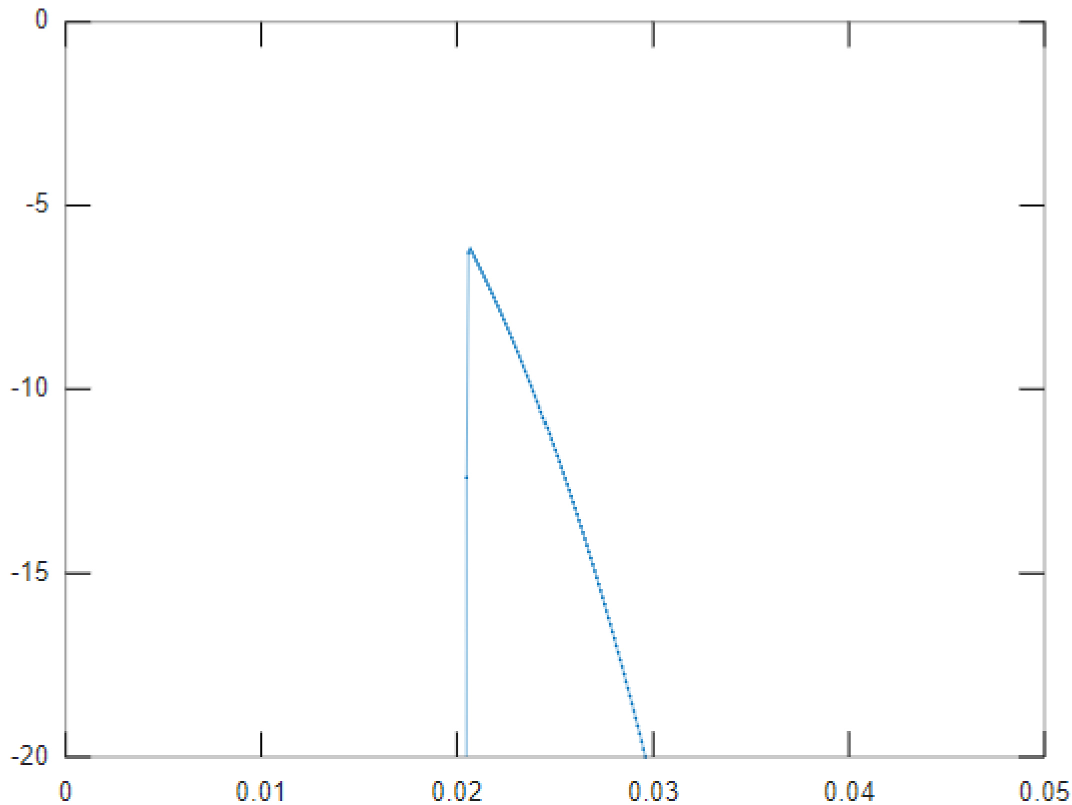
(Exercise 2d) I spent some time using Octave to try and find values of n, p so that the lower bound

$$P(n,p) = 1 - \binom{n}{s}(1-p)^{\binom{s}{2}} - \frac{1}{3}(n-1)(n-2)p^3$$

is positive. In the graph below, the horizontal axis is the p value, and the vertical axis is $P(n,p)$ where n is fixed. Each of the curves represents a single value of n, ranging from $n = 18$ to $n = 420$. The curve that attains a maximum near 0.8 is the $n = 18$ case, and the value of n increases by 6 as you jump from curve to curve going to the left.



I also plotted such graphs for n up to 1500, and the visual pattern continues: the graphs get steeper, and bunch up more and more. For $n = 1500$, the maximum value attained by $P(n,p)$ on $[0,1]$ is around -6. The graph below depicts a zoomed in view of the graph of $P(1500, p)$ for $p \in [0, 0.05]$.



Unfortunately, something probably to do with the numbers getting too large prevented me from successfully plotting these graphs for $n > 1500$. This isn't too surprising, because such computations involve expressions like $\binom{1500}{250}$ and $(1-p)^{\binom{250}{2}}$.

It's really hard to visually estimate from these graphs how large n one would need for $P(n,p) > 0$, or even if such n exists. On the other hand, from my example in part (e), we know that $n = 22$ is sufficient have a triangle free subgraph H with $|H| = \frac{n}{2}$ and $\chi(H) = 4$, so the estimate is quite weak.

Proposition 0.6 (Exercise 3a)**.** *For fixed $p \in (0,1)$ and fixed $k \in \mathbb{Z}_{\geq 1}$ and $G \in \mathcal{G}(n,p)$,*

$$\mathbb{P}(G \textit{ is k-connected}) \to 1 \qquad \textit{as } n \to \infty$$

Proof. First, observe that

$$\begin{aligned}\mathbb{P}(G \text{ is k-connected}) &= 1 - \mathbb{P}(G \text{ is not k-connected}) \\ &\geq 1 - \mathbb{P}(\exists S = \{v_1, \ldots, v_k\}, G \setminus S \text{ is not connected by paths of length 2})\end{aligned}$$

We will give an upper bound for the probability of such an S existing, which will tend to zero as $n \to \infty$.

$$\begin{aligned}
\mathbb{P}(\text{failure}) &= \mathbb{P}(\exists S = \{v_1, \dots, v_k\}, G \setminus S \text{ is not connected by paths of length 2}) \\
&= \mathbb{P}(\exists S = \{v_1, \dots, v_k\}, \exists a, b \in G \setminus S, a, b \text{ not connected by a path of length 2}) \\
&= \mathbb{P}(\exists S = \{v_1, \dots, v_k\}, \exists a, b \in G \setminus S, \forall c \in G \setminus S, ac \notin G \text{ or } bc \notin G) \\
&\leq \sum_{S=\{v_1,\dots,v_k\}} \sum_{a,b \in G\setminus S} \prod_{c \in G \setminus S} \mathbb{P}(ac \notin G \text{ or } bc \notin G) \\
&= \sum_{S=\{v_1,\dots,v_k\}} \sum_{a,b \in G\setminus S} \prod_{c \in G \setminus S} (1-p^2) \\
&= \binom{n}{k}\binom{n-k}{2}(1-p^2)^{n-k} \\
&= cn^{k+2}(1-p^2)^n
\end{aligned}$$

Since $(1-p^2) < 1$, and exponential decay always beats polynomial growth, this tends to zero as $n \to \infty$. □

Proposition 0.7 (Exercise 3b)**.** *For fixed $k \in \mathbb{Z}_{\geq 1}$. Let $p(n)$ be*

$$p(n) = \left(1 - \left(\frac{1}{\binom{n}{k}\binom{n-k}{2}n}\right)^{\frac{1}{n-k}}\right)^{\frac{1}{2}}$$

Then $p(n) \to 0$ as $n \to \infty$, and for $G \in \mathcal{G}(n, p(n))$,

$$\mathbb{P}(G \text{ is } k\text{-connected}) \to 1 \qquad \text{as } n \to \infty$$

Proof. First, we verity that $p(n) \to 0$ as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{1}{\binom{n}{k}\binom{n-k}{2}n}\right)^{\frac{1}{n-k}} &= \lim_{n\to\infty} \left(\frac{1}{\left(\frac{1}{k!}n^k\right)\left(\frac{1}{2}(n-k)^2\right)n}\right)^{\frac{1}{n-k}} \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{k}{n-k}} \left(\frac{1}{n-k}\right)^{\frac{2}{n-k}} \left(\frac{2k!}{n}\right)^{\frac{1}{n-k}}
\end{aligned}$$

Since the three factors on the right each go to 1 in the limit, this limit is 1. Thus $p(n) \to 0$ as $n \to \infty$. Now we verify the that the probability of G being connected goes to 1. From part (a), we have the estimate

$$\mathbb{P}(G \text{ is not k-connected}) \leq \binom{n}{k}\binom{n-k}{2}(1-p^2)^{n-k} = \frac{1}{n}$$

Note that $p(n)$ was chosen precisely so that this would simplify to $\frac{1}{n}$. Since $\frac{1}{n} \to 0$ as $n \to \infty$, the probability that G is k-connected goes to 1. □